A CONSTRUCTION OF CRITICAL GJMS OPERATORS USING WODZICKI'S RESIDUE

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Abstract. For an even dimensional, compact, conformal manifold without boundary we construct a conformally invariant differential operator of order the dimension of the manifold. In the conformally flat case, this operator coincides with the critical GJMS operator of Graham-Jenne-Mason-Sparling. We use the Wodzicki residue of a pseudo-differential operator of order -2, originally defined by A. Connes, acting on middle dimension forms.
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1. Introduction

In [4] Connes uses his quantized calculus to find a conformal invariant. A central part of the explicit computation of this conformal invariant in the 4-dimensional case is the study of a trilinear functional on smooth functions over the manifold M given by the relation

$$\tau(f_0, f, h) = \mathrm{Wres}(f_0[F, f][F, h]),$$

where Wres represents the Wodzicki residue, and F is a pseudo-differential operator of order 0 acting on 2-forms over M. F is given as $2\mathcal{D}-1$ with $\mathcal{D}$ the orthogonal projection on the image of d in $\mathcal{H} = \Lambda^2(T^*M) \ominus H^2$ with H^2 the space of harmonic forms. Using the general formula for the total symbol of the product of two pseudo-differential operators, Connes computed a natural bilinear differential functional of order 4, B_4 acting on $C^\infty(M)$. This bilinear differential functional

$$B_4(f, h) = 4\Delta(\langle df, dh\rangle) - 2\Delta f \Delta h + 4\langle \nabla df, \nabla dh\rangle + 8\langle df, dh\rangle J,$$

is symmetric: $B_4(f, h) = B_4(h, f)$, conformally invariant: $\widehat{B_4}(f, h) = e^{-4\eta} B_4(f, h)$, for $\widehat{g} = e^{2\eta} g$, and uniquely determined, for every f_i in $C^\infty(M)$, by the relation:

$$\tau(f_0, f_1, f_2) = \int_M f_0 B_4(f_1, f_2)\, dx.$$

On an n dimensional manifold, we use J to represent the normalized scalar curvature $2(n-1)J = \mathrm{Sc}$.

The GJMS operators [11] are invariant operators on conformal densities

$$P_{2k} : \mathcal{E}[-n/2 + k] \to \mathcal{E}[-n/2 - k]$$

with principal parts Δ^k, unless the dimension is even and $2k > n$. If n is even, the n-th order operator

$$P_n : \mathcal{E}[0] \to \mathcal{E}[-n]$$

is called *critical* GJMS operator. As noted in [11], $\mathcal{E}[0] = C^\infty(M)$ and $\mathcal{E}[-n]$ is the bundle of volume densities on M. In the 4-dimensional case, Connes has shown that

$$\begin{aligned} P_4(f) &= \Delta^2 f + 2\Delta(f)\,J + 4\langle \mathsf{P}, \nabla df\rangle + 2\langle dJ, df\rangle \\ &= \delta(d\delta + (n-2)J - 4\mathsf{P})d\,f, \end{aligned}$$

the Paneitz operator (*critical* GJMS for $n = 4$), can be derived from B_4 by the relation

$$\int_M B_4(f,h)\,dx = \frac{1}{2}\int_M f P_4(h)\,dx.$$

The GJMS operators P_{2k}, of Graham-Jenne-Mason-Sparling [11] by construction have the following properties.

o. P_{2k} exists for all k if n is odd, and if n is even exists for $1 \leq k \leq n/2$.
i. P_{2k} is formally self-adjoint.
ii. P_{2k} is conformally invariant in the sense that

$$\widehat{P}_{2k} = e^{-(n/2+k)\eta} P_{2k} e^{(n/2-k)\eta}$$

for conformally related metrics $\widehat{g} = e^{2\eta} g$.
iii. P_{2k} has a polynomial expression in ∇ and R in which all coefficients are rational in the dimension n.
iv. $P_{2k} = \Delta^k + \text{lot}$. (Here and below, lot =“lower order terms”.)
v. P_{2k} has the form

$$\delta S_{2k} d + \big(\frac{n}{2} - k\big) Q_{2k},$$

where Q_{2k} is a local scalar invariant, and S_{2k} is an operator on 1-forms of the form

$$(d\delta)^{k-1} + \text{lot} \;\text{ or }\; \Delta^{k-1} + \text{lot}\,.$$

In this last expression, d and δ are the usual de Rham operators and Δ is the form Laplacian $d\delta + \delta d$. (o), (ii), and (iv) are proved in [11]. The fact that (o) is exhaustive is proved in [9]. (i), (iii), and (v) are proved in [2].

The original work [11] uses the “ambient metric construction” of [6] to prove their existence. Since their appearance, there has been a considerable interest in constructive ways to obtain formally selfadjoint, conformally invariant powers of the Laplacian on functions with properties as stated. Examples can be found in [12] with scattering theory, in [7] with Poincaré metrics, and in [10] with tractor calculus.

In [19] (see also [20]) we have proved the following result:

Theorem 1.1. *Let M be a compact conformal manifold without boundary. Let S be a pseudo-differential operator of order 0 acting on sections of a vector bundle over M such that $S^2 f_1 = f_1 S^2$ and the pseudo-differential operator $P = [S, f_1][S, f_2]$ is conformal invariant for any $f_i \in C^\infty(M)$. There exists a unique, symmetric, and conformally invariant, bilinear, differential functional $B_{n,S}$ of order n such that*

$$\mathrm{Wres}(f_0[S, f_1][S, f_2]) = \int_M f_0 B_{n,S}(f_1, f_2)\, dx$$

for all $f_i \in C^\infty(M)$.

This result, aiming to extend part of the work of Connes [4], is based on the study of the formula for the total symbol $\sigma(P_1 P_2)$ of the product of two pseudo-differential operators P_1 and P_2 given by

$$\sum \frac{1}{\alpha!} \partial_\xi^\alpha(\sigma(P_1))\, D_x^\alpha(\sigma(P_2))$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, $\alpha! = \alpha_1! \cdots \alpha_n!$ and $D_x^\alpha = (-i)^{|\alpha|}\partial_x^\alpha$. This expression for $\sigma(P_1 P_2)$ is highly asymmetric on ∂_ξ and ∂_x. Even though, some control can be achieved when P_1 say, is given as a "multiplication" operator.

If F is the operator $2\mathcal{D} - 1$ where $\mathcal{D}$ is the orthogonal projection on the image of d inside the Hilbert space of square integrable forms of middle dimension without the harmonic ones, $\mathcal{H} = L^2(M, \Lambda_{\mathbb{C}}^{n/2} T^*M) \ominus H^{n/2}$ as defined in [4] then, this differential functional $B_n = B_{n,F}$ provides a way of constructing operators like the critical GJMS operators on compact conformal manifolds of even dimension, out of the Wodzicki residue of a commutator operator acting on middle dimension forms. We say here 'like' because up to date, we do not know the relation in between the operators we construct and the "ambient metric construction" of [6]. The following is the main result of this paper:

Theorem 1.2. *Let M be an even dimensional, compact, conformal manifold without boundary and let $(\mathcal{H}, F)$ be the Fredholm module associated to M by Connes [4]. Let P_n be the differential operator given by the relation*

$$\int_M B_n(f, h)\, dx = \int_M f P_n(h)\, dx \tag{1}$$

for all $f, h \in C^\infty(M)$. Then,

i. *P_n is formally selfadjoint.*
ii. *P_n is conformally invariant in the sense $\widehat{P_n}(h) = e^{-n\eta} P_n(h)$, if $\widehat{g} = e^{2\eta} g$.*
iii. *P_n is expressible universally as polynomial in the ingredients ∇ and R with coefficients rational in n.*
iv. *$P_n(h) = c_n \Delta^{n/2}(h) + \mathrm{lot}$ with c_n a universal constant and* lot *meaning "lower order terms".*

v. *P_n has the form $\delta S_n d$ where S_n is an operator on 1-forms given as a constant multiple of $\Delta^{n/2-1} + \mathrm{lot}$, or $(d\delta)^{n/2-1} + \mathrm{lot}$.*
vi. *P_n and B_n are related by:*

$$P_n(fh) - fP_n(h) - hP_n(f) = -2B_n(f,h).$$

The structure of this paper is as follows. In Section 2 we provide the setting, definitions, and conventions we will use throughout the rest of the paper plus a small explanation of the origin of $\mathrm{Wres}(f_0[F,f][F,h])$. In Section 3 we review Theorem 1.1 to make this paper as self contained as possible. In Section 4, by a recursive computation of the symbol expansion of F, we show that $B_n(f,h)$ has a universal expression as a polynomial on ∇, R, df, and dh. In Section 5 we prove Theorem 1.2. In [21] we will present the computations conducting to the explicit expression in the 6-dimensional case together with other calculations related to subleading symbols of pseudo-differential operators.

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2. Notations and conventions

In this paper, R represents the Riemann curvature tensor, $\mathrm{Rc}_{ij} = R^k{}_{ikj}$ represents the Ricci tensor, and $\mathrm{Sc} = \mathrm{Rc}^i{}_i$ the scalar curvature. $\mathsf{P} = (\mathrm{Rc} - Jg)/(n-2)$ is the Schouten tensor. The relation between the Weyl tensor and the Riemann tensor is given by

$$W^i{}_{jkl} = R^i{}_{jkl} + \mathsf{P}_{jk}\delta_K{}^i{}_l - \mathsf{P}_{jl}\delta_K{}^i{}_k + \mathsf{P}^i{}_l g_{jk} - \mathsf{P}^i{}_k g_{jl}$$

where δ_K represents the Kronecker delta tensor. If needed, we will "raise" and "lower" indices without explicit mention, for example, $g_{mi}R^i{}_{jkl} = R_{mjkl}$.

A differential operator acting on $C^\infty(M)$ is said to be natural if it can be written as a universal polynomial expression in the metric g, its inverse g^{-1}, the connection ∇, and the curvature R; using tensor products and contractions. The coefficients of such natural operators are called natural tensors. If M has a metric g and if D is a natural differential operator of order d, then D is said to be conformally invariant with weight w if after a conformal change of the metric $\widehat{g} = e^{2\eta}g$, D transforms as $\widehat{D} = e^{-(w+d)\eta}De^{w\eta}$. In the particular case in which $w = 0$ we say that D is conformally invariant and the previous reduces to $\widehat{D} = e^{-d\eta}D$. For a natural bilinear differential operator B of order n, conformally invariance with bi-weight (w_1, w_2) means

$$\widehat{B}(f,h) = e^{-(w_1+w_2+n)\eta}B(e^{w_1\eta}f, e^{w_2\eta}h) \tag{2}$$

for every $f, h \in C^\infty(M)$. In this work will consider only the case $w_1 = w_2 = 0$. In this particular situation, because $\widehat{dx} = e^{n\eta} dx$ we have

$$\widehat{B}(f,h)\,\widehat{dx} = B(f,h)\,dx. \tag{3}$$

Some times, when the context allows us to do so, we will abuse of the language and say that B is conformally invariant making reference to (3) instead of (2).

When working with a pseudo-differential operator P of order k, its total symbol (in some given local coordinates) will be represented as a sum of $r \times r$ matrices of the form $\sigma(P) \sim \sigma_k^P + \sigma_{k-1}^P + \sigma_{k-2}^P + \cdots$, where r is the rank of the vector bundle E on which P is acting. It is important to note that the different σ_j^P are defined only in local charts and are not diffeomorphism invariant [16]. However, Wodzicki [23] has shown that the term σ_{-n}^P enjoys a very special significance. For a pseudo-differential operator P, acting on sections of a bundle E over a manifold M, there is a 1-density on M expressed in local coordinates by

$$\operatorname{wres}(P) = \int_{||\xi||=1} \operatorname{trace}\big(\sigma_{-n}^P(x,\xi)\big)\, d\xi\, dx, \tag{4}$$

where $\sigma_{-n}^P(x,\xi)$ is the component of order $-n$ in the total symbol of P, $||\xi|| = 1$ means the Euclidean norm of the coordinate vector $(\xi_1, \cdots, \xi_n)$ in $\mathbb{R}^n$, and $d\xi$ is the normalized volume on $\{\, ||\xi|| = 1\,\}$. This *Wodzicki residue density* is independent of the local representation. An elementary proof of this matter can be found in [5]. The Wodzicki residue, $\operatorname{Wres}(P) = \int \operatorname{wres}(P)$ is then independent of the choice of the local coordinates on M, the local basis of E, and defines a trace (see [23]).

For a compact, oriented manifold M of even dimension $n = 2l$, endowed with a conformal structure, there is a canonically associated Fredholm module $(\mathcal{H}, F)$ [4]. $\mathcal{H}$ is the Hilbert space of square integrable forms of middle dimension with an extra copy of the harmonic forms $\mathcal{H} = L^2(M, \Lambda_{\mathbb{C}}^{n/2} T^*M) \oplus H^{n/2}$. Functions on M act as multiplication operators on $\mathcal{H}$. F is a pseudo-differential operator of order 0 acting in $\mathcal{H}$. On $\mathcal{H} \ominus H^{n/2} \ominus H^{n/2}$ it is obtained from the orthogonal projection $\mathcal{D}$ on the image of d, by the relation $F = 2\mathcal{D} - 1$. On $H^{n/2} \oplus H^{n/2}$ it only interchanges the two components. We are only concerned with the non-harmonic part of $\mathcal{H}$. From the Hodge decomposition theorem it is easy to see that in terms of a Riemannian metric compatible with the conformal structure of M, F restricted to $\mathcal{H} \ominus H^{n/2} \ominus H^{n/2}$ can be written as:

$$F = \frac{d\delta - \delta d}{d\delta + \delta d}.$$

2.1. **Why** $\operatorname{Wres}(f_0[F,f][F,h])$**?** As pointed out in [17], the quantum geometry of strings is concerned, among other ideas, with determinants of Laplacians associated to a surface with varying metrics. For a compact surface M without boundary

and metric g, it is possible to associate to its Laplacian Δ_g a determinant

$$\det{}'\Delta_g = \prod_{\lambda_j \neq 0} \lambda_j$$

where $0 \leq \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ are the eigenvalues of Δ_g.

To make sense out of this formal infinite product, some regularization procedure is needed. The key to study this expression as a function of g is the *Polyakov action* [18] which gives a formula for the variation of $\log\det' \Delta_g$ under a conformal change of g. For a Riemann surface M, a map $f = (f^i)$ from M to $\mathbb{R}^2$ and metric $g_{ij}(x)$ on M, the 2-dimensional Polyakov action is given by

$$I(f) = \frac{1}{2\pi} \int_M g_{ij}\, df^i \wedge \star df^j.$$

By considering instead of df its quantized version $[F, f]$, Connes [4] quantized the Polyakov action as a Dixmier trace:

$$\frac{1}{2\pi} \int_M g_{ij} df^i \wedge \star df^j = -\frac{1}{2} \operatorname{Tr}_\omega\big(g_{ij}[F, f^i][F, f^j]\big).$$

Because Wodzicki's residue extends uniquely the Dixmier trace as a trace on the algebra of PsDOs, this quantized Polyakov action has sense in the general even dimensional case.

Connes' trace theorem [3] states that the Dixmier trace and the Wodzicki residue of an elliptic PsDO of order $-n$ in an n-dimensional manifold M are proportional by a factor of $n(2\pi)^n$. In the 2-dimensional case the factor is $8\pi^2$ and so the quantized Polyakov action can be written as,

$$-16\pi^2 I = \operatorname{Wres}\big(g_{ij}[F, f^i][F, f^j]\big).$$

3. A bilinear differential functional

For a vector bundle E of rank r over a compact manifold without boundary M of dimension n, let S be a pseudo-differential operator of order k acting on sections of E. Consider P as the pseudo-differential operator given by the product $P = f_0[S, f][S, h]$ with each $f_0, f, h \in C^\infty(M)$. P is acting on the same vector bundle as S, where smooth functions on M act as multiplication operators. Each commutator $[S, f]$ defines a pseudo-differential operator of order $k-1$, thus P has order $2k-2$. Assuming $2k - 2 \geq -n$, in a given system of local coordinates the total symbol of P, up to order $-n$, is represented as a sum of $r \times r$ matrices of the form $\sigma^P_{2k-2} + \sigma^P_{2k-3} + \cdots + \sigma^P_{-n}$. We aim to study

$$\operatorname{Wres}(P) = \int_M \int_{||\xi||=1} \operatorname{trace}\big(\sigma^P_{-n}(x, \xi)\big)\, d\xi\, dx.$$

Unless otherwise stated, we assume a given system of local coordinates.

For the operator multiplication by f we have $\sigma(f) = fI$ with I the identity operator on the sections on which it is acting. In this way $\sigma(fP_2) = f\sigma(P_2)$ and in particular $\sigma_{-n}(fP_2) = f\sigma_{-n}(P_2)$. As a consequence we obtain the relation

$$\mathrm{wres}(f_0 P_2) = f_0\,\mathrm{wres}(P_2).$$

The proof of the following two results is similar to the proof for the case S of order 0 in Lemma 2.2 and Lemma 2.3 in [19] so we omit them in here.

Lemma 3.1. *$[S, f]$ is a pseudo-differential operator of order $k-1$ with total symbol $\sigma([S, f]) \sim \sum_{j\geq 1} \sigma_{k-j}([S, f])$ where*

$$\sigma_{k-j}([S, f]) = \sum_{|\beta|=1}^{j} \frac{D_x^\beta(f)}{\beta!} \partial_\xi^\beta \big(\sigma^S_{k-(j-|\beta|)}\big). \tag{5}$$

Lemma 3.2. *For $2k+n \geq 2$, with the sum taken over $|\alpha'|+|\alpha''|+|\beta|+|\delta|+i+j = n+2k$, $|\beta| \geq 1$, and $|\delta| \geq 1$,*

$$\sigma_{-n}([S, f][S, h]) = \sum \frac{D_x^\beta(f) D_x^{\alpha'+\delta}(h)}{\alpha'!\alpha''!\beta!\delta!} \partial_\xi^{\alpha'+\alpha''+\beta}(\sigma^S_{k-i}) \partial_\xi^\delta (D_x^{\alpha''}(\sigma^S_{k-j})).$$

Definition 3.3. *For every $f, h \in C^\infty(M)$ we define $B_{n,S}(f, h)$ as given by the relation*

$$B_{n,S}(f, h)\,dx := \mathrm{wres}([S, f][S, h]).$$

Because of Lemma 3.2, $B_{n,S}(f, h)$ is explicitly given by:

$$\int_{||\xi||=1} \sum \frac{D_x^\beta(f) D_x^{\alpha''+\delta}(h)}{\alpha'!\alpha''!\beta!\delta!}\, \mathrm{trace}\Big(\partial_\xi^{\alpha'+\alpha''+\beta}(\sigma^S_{k-i}) \partial_\xi^\delta (D_x^{\alpha'}(\sigma^S_{k-j}))\Big)\, d\xi$$

with the sum taken over $|\alpha'|+|\alpha''|+|\beta|+|\delta|+i+j = n+2k$, $|\beta| \geq 1$, and $|\delta| \geq 1$.

The arbitrariness of f_0 in the previous construction, implies that $B_{n,S}$ is uniquely determined by its relation with the Wodzicki residue of the operator $f_0[S, f][S, h]$ as stated in the following theorem.

Theorem 3.4. *There is a unique bilinear differential functional $B_{n,S}$ of order n such that*

$$\mathrm{Wres}(f_0[S, f][S, h]) = \int_M f_0 B_{n,S}(f, h)\,dx$$

for all f_0, f, h in $C^\infty(M)$. Furthermore, as the left hand side, the right hand side defines a Hochschild 2-cocycle over the algebra of smooth functions on M.

Proof. Linearity is evident and uniqueness follows from the arbitrariness of f_0. Let b denote the Hochschild coboundary operator on $C^\infty(M)$ and consider

$$\varphi(f_0, f_1, \cdots, f_r) = \mathrm{Wres}(f_0[S, f_1]\cdots[S, f_r]).$$

From the relation $[S, fh] = [S, f]h + f[S, h]$ we have:

$$\begin{aligned}
(b\varphi)(f_0, f_1, \cdots, f_r) &= \operatorname{Wres}(f_0 f_1 [S, f_2] \cdots [S, f_r]) \\
&\quad + \sum_{j=1}^{r-1} (-1)^j \operatorname{Wres}(f_0 [S, f_1] \cdots [S, f_j f_{j+1}] \cdots [S, f_r]) \\
&\quad + (-1)^r \operatorname{Wres}(f_r f_0 [S, f_1] \cdots [S, f_{r-1}]) \\
&= \operatorname{Wres}(f_0 f_1 [S, f_2] \cdots [S, f_r]) \\
&\quad + \sum_{j=2}^{r} (-1)^{j-1} \operatorname{Wres}(f_0 [S, f_1] \cdots [S, f_{j-1}] f_j [S, f_{j+1}] \cdots [S, f_r]) \\
&\quad + \sum_{j=1}^{r-1} (-1)^j \operatorname{Wres}(f_0 [S, f_1] \cdots [S, f_{j-1}] f_j [S, f_{j+1}] \cdots [S, f_r]) \\
&\quad + (-1)^r \operatorname{Wres}(f_r f_0 [S, f_1] \cdots [S, f_{r-1}]) \\
&= \operatorname{Wres}(f_0 f_1 [S, f_2] \cdots [S, f_r]) + (-1)^{r-1} \operatorname{Wres}(f_0 [S, f_1] \cdots [S, f_{r-1}] f_r) \\
&\quad - \operatorname{Wres}(f_0 f_1 [S, f_2] \cdots [S, f_r]) + (-1)^r \operatorname{Wres}(f_r f_0 [S, f_1] \cdots [S, f_{r-1}]) = 0
\end{aligned}$$

because of the trace property of Wres. The result follows as the particular case $r = 3$. ☐

So far, by taking $f_0 = 1$, by uniqueness, and by the trace property of Wres, we conclude

$$\int_M B_{n,S}(f, h)\, dx = \operatorname{Wres}([S, f][S, h]) = \operatorname{Wres}([S, h][S, f]) = \int_M B_{n,S}(h, f)\, dx.$$

From Definition 3.3, $B_{n,S}(f, h)\, dx = \operatorname{wres}([S, f][S, h])$ but in general wres is not a trace, hence asserting that $B_{n,S}(f, h)$ is symmetric is asserting that

$$\operatorname{wres}([S, f][S, h]) = \operatorname{wres}([S, h][S, f]).$$

To conclude the symmetry of $B_{n,S}(f, h)$ on f and h, it is necessary to request more properties on the operator S. It is enough to have the property $S^2 f = f S^2$, for every $f \in C^\infty(M)$. The symmetry of $B_{n,S}$ follows from the trace properties of the Wodzicki residue and the commutativity of the algebra $C^\infty(M)$. For a proof of the following result see [19].

Theorem 3.5. *If $S^2 f = f S^2$ for every $f \in C^\infty(M)$ then the differential functional $B_{n,S}$ in Theorem 3.4 is symmetric in f and h.*

In the most of the present work we will use the particular case $S^2 = I$.

3.1. **The functional on conformal manifolds.** If we endowed the manifold M of a conformal structure, and ask the operator $P = [S, f_1][S, f_2]$ to be independent of the metric in the conformal class, then we can say a little more about this differential functional $B_{n,S}$.

Theorem 3.6. *Let M be a compact conformal manifold without boundary. Let S be a pseudo-differential operator acting on sections of a vector bundle over M such that $S^2 f_1 = f_1 S^2$ and the pseudo-differential operator $P = [S, f_1][S, f_2]$ is conformal invariant for any $f_i \in C^\infty(M)$. There exists a unique, symmetric, and conformally invariant bilinear differential functional $B_{n,S}$ of order n such that*

$$\mathrm{Wres}(f_0[S, f_1][S, f_2]) = \int_M f_0 B_{n,S}(f_1, f_2)\, dx$$

for all $f_i \in C^\infty(M)$. Furthermore, $\int_M f_0 B_{n,S}(f_1, f_2)\, dx$ defines a Hochschild 2-cocycle on the algebra of smooth functions on M.

Proof. Uniqueness follows from (3.4). Symmetry follows from Theorem 3.5 and its conformal invariance, $\widehat{B_{n,S}} = e^{-n\eta} B_{n,S}$, follows from its construction. Indeed, the only possible metric dependence is given by the operator P. Roughly speaking, the cosphere bundle $\mathbb{S}^*M$, as a submanifold of T^*M, depends on a choice of metric in M. But since $\sigma_{-n}(P)$ is homogeneous of degree $-n$ in ξ, $\{||\xi|| = 1\}$ can be replaced by any sphere with respect to a chosen Riemannian metric on M, $\{||\xi||_g = 1\}$. The formula for the change of variable shows that

$$\int_{||\xi||=1} \mathrm{trace}(\sigma_{-n}(P)) d\xi$$

does not change within a given conformal class except trough a metric dependence of P (this argument is taken from the proof of Theorem 2.3 [22]). The Hochschild cocycle property follows from Theorem 3.4. ☐

We restrict ourselves to the particular case of an even dimensional, compact, oriented, conformal manifold without boundary M, and (E, S) given by the canonical Fredholm module $(\mathcal{H}, F)$ associated to M by Connes [4], the pseudo-differential operator F of order 0 is given by $F = (d\delta - \delta d)(d\delta + \delta d)^{-1}$ acting on the space $L^2(M, \Lambda^l_{\mathbb{C}} T^*M) \ominus H^{n/2}$, with $H^{n/2}$ the finite dimensional space of middle dimension harmonic forms. By definition F is selfadjoint and such that $F^2 = 1$. We relax the notation by denoting $B_n = B_{n,F}$ in this particular situation. P is the pseudo-differential operator of order -2 given by the product $P = f_0[F, f_1][F, f_2]$ with each $f_i \in C^\infty(M)$. P is acting on middle dimension forms. In this situation, Theorem 3.6 is stated as follows:

Theorem 3.7. *If M is an even dimensional, compact, conformal manifold without boundary and $(\mathcal{H}, F)$ is the Fredholm module associated to M by Connes* [4]

then, there is a unique, symmetric, and conformally invariant bilinear differential functional B_n of order n such that

$$\mathrm{Wres}(f_0[F,f_1][F,f_2]) = \int_M f_0 B_n(f_1,f_2)\,dx$$

for all $f_i \in C^\infty(M)$. Furthermore, $\int_M f_0 B_n(f_1,f_2)\,dx$ defines a Hochschild 2-cocycle on the algebra of smooth functions on M.

Proof. We must only prove its conformal invariance. In Lemma 2.9 [19] we show that the Hodge star operator restricted to middle dimension forms is conformally invariant in fact, acting on k-forms we have $\widehat{\star} = e^{(2k-n)\eta}\star$ for $\widehat{g} = e^{2\eta}g$. The space $\Lambda^{n/2}T^*M$ of middle dimension forms has an inner product

$$\langle \xi_1, \xi_2 \rangle = \int_M \overline{\xi_1} \wedge \star\xi_2$$

which is unchanged under a conformal change of the metric, that is, its Hilbert space completion $L^2(M, \Lambda^{n/2}T^*M)$ depends only on the conformal class of the metric. Furthermore, the Hodge decomposition:

$$\begin{aligned}\Lambda^{n/2}T^*M &= \Delta(\Lambda^{n/2}T^*M) \oplus H^{n/2} \\ &= d\delta(\Lambda^{n/2}T^*M) \oplus \delta d(\Lambda^{n/2}T^*M) \oplus H^{n/2},\end{aligned}$$

is preserved under conformal change of the metric. Indeed, the space of middle dimension harmonic forms given as $H^{n/2} = \mathrm{Ker}(d) \cap \mathrm{Ker}(d\star)$ is conformally invariant. Now if ω is a middle dimension form orthogonal to the space of harmonic forms we have $\omega = (d\widehat{\delta} + \widehat{\delta}d)\omega_1 = (d\delta + \delta d)\omega_2$ with ω_1, ω_2 middle dimension forms. It is not difficult to check that for a k-form ξ we have:

$$\widehat{\delta}\xi = e^{(2(k-1)-n)\eta}\delta e^{(-2k+n)\eta}\xi,$$

so we must have

$$d\delta\omega_2 + \delta d\omega_2 = \omega = (d\widehat{\delta} + \widehat{\delta}d)\omega_1 = d(e^{-2\eta}\delta\omega_1) + \delta(e^{-2\eta}d\omega_1)$$

which implies

$$\omega_0 := \underbrace{d(e^{-2\eta}\delta\omega_1 - \delta\omega_2)}_{\in (\mathrm{Im}\,\delta d)^\perp} = \underbrace{\delta(d\omega_2 - e^{-2\eta}d\omega_1)}_{\in (\mathrm{Im}\,d\delta)^\perp} \in H^{n/2}.$$

But also ω_0 is orthogonal to any harmonic form so $\omega_0 = 0$ and hence

$$d\widehat{\delta}\omega_1 = d\delta\omega_2 \qquad \text{and} \qquad \widehat{\delta}d\omega_1 = \delta d\omega_2.$$

We are concerned with F acting on $\Delta(\Lambda^{n/2}) = \widehat{\Delta}(\Lambda^{n/2})$. If

$$\omega = d\widehat{\delta}\omega_1 + \widehat{\delta}d\omega_1 = d\delta\omega_2 + \delta d\omega_2,$$

then

$$\widehat{F}\omega = d\widehat{\delta}\omega_1 - \widehat{\delta}d\omega_1 = d\delta\omega_2 - \delta d\omega_2 = F\omega$$

thus $\widehat{F} = F$.

Last, $\mathrm{Wres}([F,f][F,h])$ does not depend on the choice of the metric in the conformal class and the result follows. $\square$

3.2. **The functional in the flat case.** In the particular case of a flat metric, we have $\sigma^F_{-k} = 0$ for all $k \geq 1$ and so the symbol of F coincides with its principal symbol given by

$$\sigma^F_0(x,\xi) = (\varepsilon_\xi \iota_\xi - \iota_\xi \varepsilon_\xi)||\xi||^{-2}$$

for all $(x,\xi) \in T^*M$, where ε_ξ denotes exterior multiplication and ι_ξ denotes interior multiplication. Using this result, it is possible to give a formula for B_n in the flat case using the Taylor expansion of the function

$$\psi(\xi,\eta) = \mathrm{trace}\big(\sigma^F_0(\xi)\sigma^F_0(\eta)\big).$$

Here $\xi,\eta \in T^*_xM \smallsetminus \{0\}$ and $\sigma^F_0(\xi)\sigma^F_0(\eta)$ is acting on middle-forms.

Proposition 3.8. *With $\sigma^F_0(\xi)\sigma^F_0(\eta)$ acting on m-forms we have:*

$$\psi(\xi,\eta) = \mathrm{trace}(\sigma^F_0(\xi)\sigma^F_0(\eta)) = a_{n,m}\frac{\langle\xi,\eta\rangle^2}{||\xi||^2||\eta||^2} + b_{n,m},$$

where

$$b_{n,m} = \binom{n-2}{m-2} + \binom{n-2}{m} - 2\binom{n-2}{m-1} = \binom{n}{m} - a_{n,m}.$$

For the proof see Theorem 4.3 [19]. For the case $n=4$ and $m=2$ we obtain

$$\psi(\xi,\eta) = 8\langle\xi,\eta\rangle^2(||\xi||||\eta||)^{-2} - 2.$$

Here there is a discrepancy with Connes, he has $4\langle\xi,\eta\rangle^2(||\xi||||\eta||)^{-2} +$ constant. For $n=6$, $m=3$,

$$\psi(\xi,\eta) = 24\langle\xi,\eta\rangle^2(||\xi||||\eta||)^{-2} - 4.$$

In the flat case, with $S = F$, (5) reduces to:

$$\sigma_{-r}([F,f]) = \sum_{|\beta|=r}\frac{D^\beta_x f}{\beta!}\partial^\beta_\xi(\sigma^F_0).$$

Using this information we deduce from Lemma 3.2 with $\alpha' = 0$, $\alpha'' = \alpha$, and by the definition of $B_{n,S}$:

$$B_{n\,\mathrm{flat}}(f,h) = \int_{||\xi||=1}\sum\frac{D^\beta_x f D^{\alpha+\delta}_x h}{\alpha!\beta!\delta!}\,\mathrm{trace}\Big(\partial^{\alpha+\beta}_\xi(\sigma^F_0)\partial^\delta_\xi(\sigma^F_0)\Big)\,d\xi$$

with the sum taken over $|\alpha|+|\beta|+|\delta| = n$, $1 \leq |\beta|$, $1 \leq |\delta|$.

Definition 3.9. *We denote by $T'_n\psi(\xi,\eta,u,v)$ the term of order n in the Taylor expansion of $\psi(\xi,\eta)$ minus the terms with only powers of u or only powers of v. That is to say,*

$$T'_n\psi(\xi,\eta,u,v) := \sum \frac{u^\beta}{\beta!}\frac{v^\delta}{\delta!}\operatorname{trace}\bigl(\partial_\xi^\beta(\sigma_0^F(\xi))\partial_\eta^\delta(\sigma_0^F(\eta))\bigr), \tag{6}$$

with the sum taken over $|\beta|+|\delta|=n$, $|\beta|\geq 1$, and $|\delta|\geq 1$.

There is an explicit expression for B_n in the flat case in terms of the Taylor expansion of $\psi(\xi,\eta)$:

Theorem 3.10.
$$B_{n\,\mathrm{flat}}(f,h) = \sum A_{a,b}(D_x^a f)(D_x^b h),$$
where
$$\sum A_{a,b}u^a v^b = \int_{||\xi||=1}\bigl(T'_n\psi(\xi,\xi,u+v,v) - T'_n\psi(\xi,\xi,v,v)\bigr)\,d\xi$$
with $T'_n\psi(\xi,\eta,u,v)$ is given by (6).

For details and a proof of the previous theorem see Section 4 [19]. In the 4 dimensional flat case, we obtain with Maple:

$$B_{4,\mathrm{flat}}(f,h) = -4\Bigl(f_{;ij}{}^{j}h_{;}{}^{i} + h_{;ij}{}^{j}f_{;}{}^{i}\Bigr) - 4f_{;ij}h_{;}{}^{ij} - 2f_{;i}{}^{i}h_{;j}{}^{j},$$

and in the 6 dimensional flat case:

$$\begin{aligned}&B_{6\,\mathrm{flat}}(df,dh)\\ &= 12\,(f_{;i}h_{;}{}^{i}{}_{j}{}^{j}{}_{k}{}^{k} + h_{;i}f_{;}{}^{i}{}_{j}{}^{j}{}_{k}{}^{k}) + 24\,(f_{;ij}h_{;}{}^{ij}{}_{k}{}^{k} + h_{;ij}f_{;}{}^{ij}{}_{k}{}^{k})\\ &\quad + 6\,(f_{;i}{}^{i}h_{;j}{}^{j}{}_{k}{}^{k} + h_{;i}{}^{i}f_{;j}{}^{j}{}_{k}{}^{k}) + 24\,f_{;ij}{}^{j}h_{;}{}^{i}{}_{k}{}^{k} + 16f_{;ijk}h_{;}{}^{ijk},\end{aligned}$$

Here each summand is explicitly symmetric on f and h.

Remark 3.11. *By looking at the construction of B_n in the flat case and by Theorem 3.10, we know that the coefficients of $B_{n\,\mathrm{flat}}$ are obtained from the Taylor expansion of the function $\psi(\xi,\xi)$ and integration on the sphere $||\xi||=1$. Hence they are universal in the flat case. Furthermore, since the volume of the sphere $\mathbb{S}^{n-1}$ is $2\pi^{n/2}/\Gamma(n/2)$, any integral of a polynomial in the ξ_i's with rational coefficients will be a rational multiple of $\pi^{n/2}$. To avoid the factor $\pi^{n/2}$, we have assumed in* (4) *$d\,\xi$ to be normalized. We conclude from Theorem 3.10 that the coefficients of $B_{n\,\mathrm{flat}}(f,h)$ are all rational in n.*

In the particular case in which $\widehat{g} = e^{2\eta}g$ with g the flat metric on M, we have the conformal change equation for the Ricci tensor:

$$\eta_{;ij} = -\mathsf{P}_{ij} - \eta_{;i}\,\eta_{;j} + \frac{1}{2}\eta_{;k}\,\eta_{;}{}^{k}\,g_{ij}, \tag{7}$$

which allows to replace all higher derivatives on η with terms with the Ricci tensor. In this way, using (7) it is possible to obtain the expression for B_n in the conformally flat case out of its expression in the flat metric. We exemplified this process in [20] for the six dimensional case.

Remark 3.12. *The universality of the expression for $B_{n\,\mathrm{flat}}$ is preserved into the universality of the expression for B_n in the conformally flat case, $B_{n\,\mathrm{conf.\,flat}}$. In the next section we will see in general that there is a universal expression for B_n. Furthermore, using* (7) *we conclude that also the coefficients of $B_{n\,\mathrm{conf.\,flat}}$ are rational in n.*

4. A recursive approach to the symbol of F

In this section we give a recursive way of computing $\sigma_{-j}(F)$ in some given local charts, by considering $F = 2\mathcal{D} - 1$ with $\mathcal{D}$ the orthogonal projection on the image of d inside $L^2(M, \Lambda^{n/2}_{\mathbb{C}} T^*M)$. Both F and $\mathcal{D}$ are pseudo-differential operators of order 0 with $\sigma_0(F) = 2\sigma_0(\mathcal{D}) - I$ and $\sigma_j(F) = 2\sigma_j(\mathcal{D})$ for each $j < 0$. We shall use the symbols of the differential operators of order 2, $d\delta$ and Δ. The purpose of this section is to understand the relation in between B_n and the curvature tensor.

Remark 4.1. *Even though the different $\sigma_{-j}(F)$ are not diffeomorphic invariant, an explicit understanding of their expressions in a given coordinate chart can be of optimal use, an example is present in* [15] *and* [14] *where using normal coordinates and recursive computation of symbols they were able to prove that the action functional, that is, the Wodzicki residue of $\Delta^{-n/2+1}$, is proportional to the integral of the scalar curvature by a constant depending on n :*

$$\operatorname{Wres} \Delta^{-n/2+1} = \frac{(n/2-1)\Omega_n}{6} \int_M \operatorname{Sc} |v_g|,$$

with Ω_n the volume of the standard n-sphere, and $|v_g|$ the 1-density associated to the normalize volume form of M.

From Lemma 2.4.2 [8], it is possible to understand the relation between the σ_j^{Δ} and the metric at a given point x. The same reasoning can be applied to the operator $d\delta - \delta d$ and by addition the same is true for $d\delta$. We represent $\sigma(\Delta)$, $\sigma(d\delta - \delta d)$, and $\sigma(d\delta)$ as $\sigma(\Delta) = \sigma_2^{\Delta} + \sigma_1^{\Delta} + \sigma_0^{\Delta}$, $\sigma(d\delta - \delta d) = \sigma_2^{d\delta-\delta d} + \sigma_1^{d\delta-\delta d} + \sigma_0^{d\delta-\delta d}$, and $\sigma(d\delta) = \sigma_2^{d\delta} + \sigma_1^{d\delta} + \sigma_0^{d\delta}$ with each σ_j the component of homogeneity j on the co-variable ξ. The conclusion reads: σ_2 only invokes $g_{ij}(x)$, σ_1 is linear in the first partial derivatives of the metric at x with coefficients depending smoothly on the $g_{ij}(x)$, and σ_0 can be written as a term linear in the second partial derivatives of the metric at x with coefficients depending smoothly on the $g_{ij}(x)$ plus a quadratic term linear in the first partial derivatives of the metric at x.

Since in terms of a Riemannian metric compatible with the conformal structure of M we have $\Delta\mathcal{D} = d\delta$ we know $\sigma(\Delta\mathcal{D}) = \sigma(d\delta)$. Thus the formula for the total

symbol of the product of two pseudo-differential operators implies

$$\sigma_2^{d\delta}+\sigma_1^{d\delta}+\sigma_0^{d\delta}=\sigma(d\delta)=\sigma(\Delta\mathcal{D})\sim\sum\frac{1}{\alpha!}\partial_\xi^\alpha\sigma(\Delta)D_x^\alpha(\sigma(\mathcal{D}))$$
$$\sim\sum\frac{1}{\alpha!}\partial_\xi^\alpha(\sigma_2^\Delta+\sigma_1^\Delta+\sigma_0^\Delta)D_x^\alpha(\sigma_0^{\mathcal{D}}+\sigma_{-1}^{\mathcal{D}}+\sigma_{-2}^{\mathcal{D}}+\cdots).$$

Expanding the right hand side into sum of terms with the same homogeneity we conclude:

Lemma 4.2. *In any given system of local charts, we can express the total symbol of $\mathcal{D}$, $\sigma(\mathcal{D})\sim\sigma_0^{\mathcal{D}}+\sigma_{-1}^{\mathcal{D}}+\cdots$ in a recursive way by the formulae:*

$$\sigma_0^{\mathcal{D}}=(\sigma_2^\Delta)^{-1}\sigma_2^{d\delta},$$
$$\sigma_{-1}^{\mathcal{D}}=(\sigma_2^\Delta)^{-1}\Big(\sigma_1^{d\delta}-\sigma_1^\Delta\sigma_0^{\mathcal{D}}-\sum_{|\alpha|=1}\partial_\xi^\alpha(\sigma_2^\Delta)D_x^\alpha(\sigma_0^{\mathcal{D}})\Big),$$
$$\sigma_{-2}^{\mathcal{D}}=(\sigma_2^\Delta)^{-1}\Big(\sigma_0^{d\delta}-\sigma_1^\Delta\sigma_{-1}^{\mathcal{D}}-\sigma_0^\Delta\sigma_0^{\mathcal{D}}$$
$$-\sum_{|\alpha|=1}\big(\partial_\xi^\alpha(\sigma_2^\Delta)D_x^\alpha(\sigma_{-1}^{\mathcal{D}})+\partial_\xi^\alpha(\sigma_1^\Delta)D_x^\alpha(\sigma_0^{\mathcal{D}})\big)-\sum_{|\alpha|=2}\frac{1}{\alpha!}\partial_\xi^\alpha(\sigma_2^\Delta)D_x^\alpha(\sigma_0^{\mathcal{D}})\Big),$$
$$\sigma_{-r}^{\mathcal{D}}=-(\sigma_2^\Delta)^{-1}\Big(\sigma_1^\Delta\sigma_{-r+1}^{\mathcal{D}}+\sigma_0^\Delta\sigma_{-r+2}^{\mathcal{D}}+\sum_{|\alpha|=1}\partial_\xi^\alpha(\sigma_2^\Delta)D_x^\alpha(\sigma_{-r+1}^{\mathcal{D}})$$
$$+\sum_{|\alpha|=1}\partial_\xi^\alpha(\sigma_1^\Delta)D_x^\alpha(\sigma_{-r+2}^{\mathcal{D}})+\sum_{|\alpha|=2}\frac{1}{\alpha!}\partial_\xi^\alpha(\sigma_2^\Delta)D_x^\alpha(\sigma_{-r+2}^{\mathcal{D}})\Big),$$

for every $r\geq 3$. Thus the total symbol of F, $\sigma^F\sim\sigma_0^F+\sigma_{-1}^F+\cdots$, can be recover from the relations $\sigma_0^F=2\sigma_0^{\mathcal{D}}-I$ and $\sigma_{-k}^F=2\sigma_{-k}^{\mathcal{D}}$ for $k\leq 1$.

Lemma 4.3. *The bilinear differential functional B_n on Theorem 3.7 has a universal expression as a polynomial on ∇, R, df, and dh with coefficients rational on n.*

Proof. By choosing the coordinates to be normal coordinates, we can assume the following: $g_{ij}(x)=\delta_{Kij}$, that the partial derivatives of the metric vanish at x, and that any higher order partial derivative of the metric at x are expressed as polynomials on ∇ and R (see for example Corollary 2.9 [13]). By Lemma 4.2 each $\sigma_{-k}^F(x,\xi)$ has a polynomial expression on ∇ and R at the point x. Every product $\partial_\xi^\alpha(\sigma_{-i}^F)\partial_\xi^\beta(D_x^\gamma(\sigma_{-j}^F))$ will have a polynomial expression on ∇ and R. These properties are preserved after integration of ξ on $||\xi||=1$ and hence

$$B_n(f,h):=\int_{||\xi||=1}\sum\frac{D_x^\beta(f)D_x^{\alpha''+\delta}(h)}{\alpha'!\alpha''!\beta!\delta!}\operatorname{trace}\Big(\partial_\xi^{\alpha'+\alpha''+\beta}(\sigma_{-i}^F)\partial_\xi^\delta(D_x^{\alpha'}(\sigma_{-j}^F))\Big)\,d\xi$$

with sum taken over $|\alpha'|+|\alpha''|+|\beta|+|\delta|+i+j=n$, $|\beta|\geq 1$, $|\delta|\geq 1$ has an expression as a polynomial in the ingredients ∇, R, df, and dh. The explicit

expression from Lemma 4.2 for each σ^F_{-k} in terms of σ^Δ_i and $\sigma^{d\delta}_j$ depends on the metric, the dimension, and the local coordinates since the different σ_j are not diffeomorphic invariant, but (4) and hence B_n is independent of local representations. We conclude that the expression we obtain for B_n is universal.

This independence of local representations allows as to conclude the rationality of the coefficients of B_n as follows. By Lemma 4.2 each $\sigma^{\mathcal{D}}_{-j}$ is a polynomial expression in the $\partial_x\sigma^\Delta_k$, $\partial_x\partial_\xi\sigma^\Delta_k$, and $\partial_x\sigma^{d\delta}_k$, with coefficients rational in n. By choosing the local coordinates to be normal coordinates, we can assure that all the expressions of these symbols as polynomials on ∇ and R are of coefficients rational in n. □

4.1. **A filtration by degree.** If we fix h (or f), $B_n(f,h)$ is a differential operator or order $n-1$ on f (or h). It is acting on smooth functions and producing smooth functions. Since $\widehat{B_n} = e^{-n\eta}B_n$ when we transform the metric conformally by $\widehat{g} = e^{2\eta}g$, we have that B_n is of level n, (see [1]) that is, the expression $B_n(f,h)$ is a sum of homogeneous polynomials in the ingredients $\nabla^\alpha f$, $\nabla^\beta h$, and $\nabla^\gamma R$ for multi-indices α, β, and γ, in the following sense, each monomial must satisfies the homogeneity condition given by the rule:

$$\text{twice the appearances of } R \ + \ \text{number of covariant derivatives} = n$$

where for covariant derivatives we count all of the derivatives on R, f, and h, and any occurrence of W, Rc, P, Sc, or J is counted as an occurrence of R. Furthermore, by the restriction $|\beta| \geq 1$ and $|\delta| \geq 1$ in Lemma 3.2, we know that $B_n(f,h)$ is made of the ingredients $\nabla^\alpha df$, $\nabla^\beta dh$, and $\nabla^\gamma R$.

By closing under addition, we denote by $\mathcal{P}_n$ the space of these polynomials. For a homogeneous polynomial p in $\mathcal{P}_n$, we denote by $\deg_R$ its degree in R and by $\deg_\nabla$ its degree in ∇. In this way, $2\deg_R + \deg_\nabla = n$, with $\deg_\nabla \geq 2$ and hence $2\deg_R \leq n-2$ for $B_n(f,h)$. We say that p is in $\mathcal{P}_{n,r}$ if p can be written as a sum of monomials with $\deg_R \geq r$, or equivalently, $\deg_\nabla \leq n-2r$. We have a filtration by degree:

$$\mathcal{P}_n = \mathcal{P}_{n,0} \supseteq \mathcal{P}_{n,1} \supseteq \mathcal{P}_{n,2} \supseteq \cdots \supseteq \mathcal{P}_{n,\frac{n-2}{2}},$$

and $\mathcal{P}_{n,r} = 0$ for $r > (n-2)/2$. There is an important observation to make. An expression which a priori appears to be in, say $\mathcal{P}_{6,1}$, may actually be in a subspace of it, like $\mathcal{P}_{6,2}$. For example,

$$\underbrace{f_{;i}h_{;jkl}W^{ijkl}}_{\in\mathcal{P}_{6,1}} = \underbrace{f_{;i}h_{;j}\mathsf{P}_{kl}W^{ikjl} + \frac{1}{2}f_{;i}h_{;j}W^{i}{}_{klm}W^{jklm}}_{\in\mathcal{P}_{6,2}}, \tag{8}$$

by reordering covariant derivatives and making use of the symmetries of the Weyl tensor.

In the particular case $n = 4$, k_R can be 0 or 1, hence B_4 can be written as

$$B_4(f,h) = \underbrace{p_4(df,dh)}_{\in \mathcal{P}_{4,0}} + \underbrace{p_{R,4}(df,dh)}_{\in \mathcal{P}_{4,1}}$$

where $p_{R,4}(df,dh)$ is a trilinear form on R, df, and dh. Explicitly,

$$p_4(df,dh) = B_{4\,\mathrm{flat}}(f,h) \text{ and } p_{R,4}(df,dh) = 8f_{;i}h_{;}{}^{i}J.$$

In the 6-dimensional case, $k_R \in \{0,1,2\}$ thus

$$B_6(f,h) = \underbrace{p_6(df,dh)}_{\in\mathcal{P}_{6,0}/\mathcal{P}_{6,1}} + \underbrace{p_{6,1}(df,dh)}_{\in\mathcal{P}_{6,1}/\mathcal{P}_{6,2}} + \underbrace{p_{6,2}(df,dh)}_{\in\mathcal{P}_{6,2}},$$

with

$$p_{6,1} = p_{6,R} + p_{6,R'} + p_{6,R''},$$

- $p_6(df,dh) = B_{6\,\mathrm{flat}}(f,h)$,
- $p_{6,R}(df,dh)$, polynomial on $\nabla^a df$, $\nabla^b dh$, and R,
- $p_{6,R'}(df,dh)$, polynomial on $\nabla^a df$, $\nabla^b dh$, and ∇R,
- $p_{6,R''}(df,dh)$, polynomial on $\nabla^a df$, $\nabla^b dh$, and $\nabla\nabla R$, and
- $p_{6,2}(df,dh)$, polynomial on $\nabla^a df$, $\nabla^b dh$, and RR.

From the previous expressions, it is evident that there exists a sub-filtration inside each $\mathcal{P}_{n,l}$ for $l \geq 1$. Such a filtration is more complicated to describe in higher dimension because of the presence of terms like $\nabla^a R \nabla^c R \cdots$. Also it is important to note that $p_n(df,dh)$ is precisely $B_{n,\mathrm{flat}}(f,h)$, i.e. the flat version of B_n coincide with the expression in the filtration without curvature terms.

5. P_n and the Wodzicki residue

Now we are ready to prove our main result:

Proof. **Of Theorem 1.2**. Since $B_n(f,h)$ is a differential functional on f and h, Stokes' theorem applied to $\int_M B_n(f,h)\,dx$ leads to the expression $\int_M f P_n(h)\,dx$, where P_n is a differential operator on h. Uniqueness of P_n follows from the arbitrariness of f.

Formally selfadjointness is a consequence of the symmetry of B_n, that is:

$$\int_M f P_n(h)\,dx = \int_M B_n(f,h)\,dx = \int_M B_n(h,f)\,dx = \int_M P_n(f) h\,dx.$$

Note that

$$\int_M f e^{-n\eta} P_n(h)\,dx = \int_M B_n(fe^{-n\eta},h)\,dx = \int_M \widehat{B_n}(fe^{-n\eta},h)\,\widehat{dx}$$
$$= \int_M fe^{-n\eta}\widehat{P_n}(h)\,\widehat{dx} = \int_M f\widehat{P}_n(h)\,dx$$

for every $f \in C^\infty(M)$. (ii) follows from the arbitrariness of f.

From Lemma 4.3 there is a universal expression for B_n and Stoke's theorem does not affect this universality nor the rationality of its coefficients, thus there is a universal expression for P_n as a polynomial on ∇ and R, with coefficients rational on n, and (iii) is proved.

Next we prove (iv). Because of the filtration, we can write $B_n(f,h)$ in the form $B_n(f,h) = p_n(df,dh) + p_{n,R}(df,dh)$ where any possible curvature term is in the $p_{n,R}$ part and those terms in p_n are of the form

$$f_{;j_1}{}^{j_1}\cdots{}_{j_s}{}^{j^s}{}_{i_1\cdots i_r}\, h_{;k_1}{}^{k_1}\cdots{}_{k_t}{}^{k_t i_1\cdots i_r} \tag{9}$$

with $2r+2s+2t=n$. Each time we apply Stokes' theorem to such a term and reorder the indices, we obtain an expression like:

$$(\pm)\int f\Delta^{n/2}(h)\,dx + \int f\,\mathrm{lot}(h,R)\,dx$$

where $\mathrm{lot}(h,R)$ represents a differential operator on h which is polynomial on ∇ and R with order on h lower than n. It follows that $P_n(h) = c_n\Delta^{n/2}(h) + \mathrm{lot}$ with c_n a universal constant and lot a sum of lower order terms.

To prove (v) we take a closer look at $B_n(f,h)$. Each time we apply Stokes' theorem to a term of the form (9) for which $r \geq 1$, we can reorder the indices in such a way that we have an expression like:

$$(\pm)\int f\Delta^{n/2}(h)\,dx + \int f\big(\mathrm{lot}(h_i,R)\big)_{;}{}^{i}\,dx$$

where now $\mathrm{lot}(h_i,R)$ represents a sum of contractions of products of the form $\nabla^\alpha h\times$ (some curvature term), for some multi-index α with i present in α. Each time we apply Stokes' theorem to a term of the form (9) for which $r=0$, we can reorder the indices in such a way that we have an expression like:

$$(\pm)\int f\Delta^{n/2}(h)\,dx + \int f\big(\mathrm{lot}(h,R)\big)_{;i}{}^{i}\,dx \tag{10}$$

where $\mathrm{lot}(h,R)$ represents a sum of lower order terms. Each time we apply Stokes' theorem to any term of $B_n(f,h)$, not of the form (9), that is to say with some curvature term, we can reorder the indices in such a way that we have an expression of the form:

$$\int f\big(\mathrm{lot}(h_i,R)\big)_{;}{}^{i}\,dx + \int f\big(\mathrm{lot}(h,R_i)\big)_{;}{}^{i}\,dx \tag{11}$$

where $\mathrm{lot}(h,R_i)$ represents a sum of contractions of products of the form $\nabla^\alpha h\times$ (some curvature term), for some multi-index α in such a way that i is present in one of the curvature terms (not in α).

Applying the Leibniz rule to the first appearance of i in the second summand of (10) produces more terms of the same form as (11). We conclude:

$$P_n(h) = c_n\Delta^{n/2}(h) + \sum\big(\mathrm{lot}(h_i,R)\big)_{;}{}^{i} + \sum\big(\mathrm{lot}(h,R_i)\big)_{;}{}^{i}.$$

Next, consider the operator S'_n acting on exact 1-forms:

$$S'_n : dh = h_{;i}\,dx^i \mapsto -\Big(\sum \mathrm{lot}(h_i, R) + \sum \mathrm{lot}(h, R_i)\Big)\,dx^i.$$

It follows that $P_n(h) = c_n\Delta^{n/2}h + \delta S'_n d(h)$. Hence $P_n = \delta S_n d$ where S_n is a constant multiple of $\Delta^{n/2-1} + \mathrm{lot}$ or $(d\delta)^{n/2-1} + \mathrm{lot}$, where any curvature term is absorbed by the lot part.

For (vi), we will prove that for any $k \in C^\infty(M)$ we have

$$\int_M k\big(P_n(fh) - fP_n(h) - hP_n(f)\big)\,dx = -2\int_M k\,B_n(f,h)\,dx \tag{12}$$

and the result will follow from the arbitrariness of k. By (1) the left hand side of (12) is equal to:

$$\begin{aligned}
&\int_M B_n(k, fh) - B_n(kf, h) - B_n(kh, f)\,dx \\
&= \mathrm{Wres}\big([F,k][F,fh] - [F,kf][F,h] - [F,kh][F,f]\big) \\
&= \mathrm{Wres}\big(FkFfh - FkfhF - kFFfh + kFfhF \\
&\qquad - FkfFh + FkfhF + kfFFh - kfFhF \\
&\qquad - FhkFf + FhkfF + hkFFf - hkFfF\big) \\
&= -2\,\mathrm{Wres}(kFfFh - kFfhF - kfFFh + kfFhF) \\
&= -2\,\mathrm{Wres}\big(k[F,f][F,h]\big) = -2\int_M k\,B_n(f,h)\,dx,
\end{aligned}$$

where we have used the commutativity of the algebra $C^\infty(M)$, the linearity and the trace property of Wres, and the property $F^2 = 1$. □

Remark 5.1. *The value of c_n in Theorem 1.2 (iii) gets determined in the flat case since no terms with curvature affect it. In the 4 dimensional case $c_4 = 2$ and in the 6 dimensional case $c_6 = -4$.*

The relation between the GJMS operators in the general even dimensional case and the P_n operators described in this work still unknown to the author at this time. Nevertheless, in the flat case, both operators coincide up to a constant multiple with a power of the Laplacian, $\Delta^{n/2}$, and both operator enjoy the same conformal property

$$\widehat{P} = e^{-n\eta}P$$

for $\widehat{g} = e^{2\eta}g$. Hence, they must also coincide, up to a constant multiple, in the conformally flat case.

Proposition 5.2. *In the even dimensional case, inside the conformally flat class of metrics, the critical GJMS operator and the operator P_n described in this work coincide up to a constant multiple.*

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